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MOTION OF A HIGHLY VISCOUS LIQUID IN A ROTATING TORUS

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We consider a motion of a viscous incompressible liquid in a toroidal cavity within a top spinning with an arbitrary angular velocity and acceleration. The results obtained can be used to determine the position of the toroidal tube filled with a viscous liquid relative to the top axes of inertia, which will minimize the time necessary to stabilize the motion of the top.

1. Statement of the problem. It was shown in [1] for the motion of a solid with cavities completely filled with a viscous liquid, that for the first approximation to the value of the Reynolds number $R = l^2 / T\nu \ll 1$ and for large values of time $t > l^2/\nu$, the contribution of the relative motion of the liquid to the moment of impulse of the solid-liquid system does not depend on the initial motion of the liquid and can be written in the form

$$\mathbf{L} = -\frac{\rho}{\nu}\sum_{i,j=1}^{3} P_{ij}\varepsilon_i(t)\,\mathbf{e}^{(j)}, \qquad P_{ij} = -\int_V \mathbf{e}^{(j)}[\mathbf{r}, \boldsymbol{\zeta}^{(i)}]\,dV \tag{1.1}$$

where the integration is performed over the volume of the cavity, ε is the angular acceleration of the solid and $\zeta^{(i)}$ is the solution of the system (see [1])

$$\Delta\boldsymbol{\zeta}^{(i)} = \nabla s^{(i)} + [\mathbf{e}^{(i)}, \mathbf{r}], \qquad \operatorname{div}\boldsymbol{\zeta}^{(i)} = 0, \qquad \boldsymbol{\zeta}^{(i)}|_S = 0 \tag{1.2}$$

When time is large, the quantities $\zeta^{(i)}$ and $s^{(i)}$ determine the velocity $\mathbf{u}$ of the liquid relative to the solid, and its generalized pressure p

$$\mathbf{u} = \frac{1}{\nu}\sum_{i=1}^{3}\varepsilon_i\boldsymbol{\zeta}^{(i)}, \qquad p = \sum_{i=1}^{3}\varepsilon_i s^{(i)}$$

In [1] we find the values of P_{ij} computed for a sphere, an ellipsoid and a cylinder. Below we consider the case of a toroidal cavity, representing the simplest example of a doubly connected region.

2. Investigation of equations of motion of a liquid in a torus. Let the cavity have the form of a torus with the median line radius denoted by R and the tube radius by r. We introduce the intrinsic coordinate system of the torus with its center at the center of symmetry, the oz-axis directed along the symmetry axis and the ox- and oy-axes situated in the plane of symmetry. In such a coordinate system the tensor $\| P_{ij} \|$ is diagonal and $P_{xx} = P_{yy}$.

Since the right-hand side of (1.2) contains the unit vector of the Cartesian system, we shall seek the Cartesian components of the velocities $\zeta^{(i)}$. We shall however write the operator Δ using the toroidal α, β, φ; $\mathrm{ch}\,\alpha \equiv \tau$ coordinate system (see e.g. [2]) which allows the separation of variables

$$x = c\rho \cos\varphi, \quad y = c\rho \sin\varphi, \quad z = c \sin\beta / A$$
$$(c^2 = R^2 - r^2, \quad A = \tau - \cos\beta, \quad \rho = \mathrm{sh}\,\alpha / A)$$

Within the torus we have $\tau \geqslant \tau_0 = R / r$ and in the following we shall assume, for simplicity, that $c = 1$.

Since $\mathrm{div}\,[\mathbf{e}^{(i)}, \mathbf{r}] = 0$, we have $\Delta s^{(i)} = 0$, i.e. the effective pressure is a harmonic function.

We seek a solution of the system (1.2) in the form $\zeta^{(i)} = \zeta_1^{(i)} + \zeta_2^{(i)}$, where $\zeta_1^{(i)}$ is given by the equation

$$\Delta\zeta_1^{(i)} = [\mathbf{e}^{(i)}, \mathbf{r}], \qquad \zeta_1^{(i)}|_S = 0 \tag{2.1}$$

while $\zeta_2^{(i)}$ can be found using $\zeta_1^{(i)}$, from

$$\Delta\zeta_2^{(i)} = \nabla s^{(i)}, \quad \mathrm{div}\,\zeta_2^{(i)} = -\,\mathrm{div}\,\zeta_1^{(i)}, \quad \zeta_2^{(i)}|_S = 0 \tag{2.2}$$

so that $\Delta\,\mathrm{rot}\,\zeta_2^{(i)} = 0$.

From (2.1) and (2.2) it follows that $\Delta\,\mathrm{div}\,\zeta_2^{(i)} = -\Delta\,\mathrm{div}\,\zeta_1^{(i)} = 0$, i.e. $\mathrm{div}\,\zeta_1^{(i)}$, $\mathrm{div}\,\zeta_2^{(i)}$ and $\mathrm{rot}\,\zeta_2^{(i)}$ are also harmonic functions. Below we shall solve (2.1) and (2.2) for a partcular set of the unit vectors $\mathbf{e}^{(i)}$.

3. Angular acceleration in the direction of the oz-axis. Let us first consider the case $i = 3$. We write the equations in the following form (omitting the superscript in $\zeta^{(i)}$ for simplicity):

$$\Delta\zeta_{1x} = -y, \quad \zeta_{1x}|_S = 0, \quad \Delta\zeta_{1y} = x, \quad \zeta_{1y}|_S = 0, \quad \Delta\zeta_{1z} = 0, \quad \zeta_{1z}|_S = 0 \tag{3.1}$$

This implies that $\zeta_{1z} \equiv 0$. The boundary conditions for ζ_{1x} and ζ_{1y} are independent of β and φ, while the right-hand sides of the corresponding equations are even in β and contain φ only in the form of the multiplying factors $\sin\varphi$ and $\cos\varphi$. Taking this into account and using the method of separation of variables, we can write the solution of (3.1) in the form

$$\zeta_{1x} = -\zeta_1 \sin\varphi, \qquad \zeta_{1y} = \zeta_1 \cos\varphi \tag{3.2}$$

$$\zeta_1 = A^{1/2}\Big[\frac{1}{2} f_0(\tau) + \sum_{m=1}^{\infty} f_m(\tau) \cos m\beta\Big]$$

$$f_m(\tau) = b_m Q^1_{m-1/2}(\tau) + \frac{8\sqrt{2}}{15\pi}\,\frac{1}{\sqrt{\tau^2 - 1}}\,Q^2_{m+1/2}(\tau)$$

where $Q_n^k(\tau)$ are the associated Legendre functions of second kind (see e.g. [3]) and the coefficients b_m are determined by the boundary conditions $f_m(\tau_0) = 0$. If τ_0 is large, then the lower approximation gives e.g. $b_0 = (\sqrt{2} / \pi)\tau_0^{-2}$.

Next we consider ζ_2. From (3.2) it follows that $\mathrm{div}\,\zeta_1 = 0$, therefore for ζ_2 we have $\Delta\zeta_2 = \nabla s$, $\mathrm{div}\,\zeta_2 = 0$ and $\zeta_2|_S = 0$, which implies that $\zeta_2 \equiv 0$ and $s \equiv 0$.

Thus the expressions (3.2) give an exact solution of (1.2) for $i = 3$. This means that the angular acceleration directed along the torus axis is associated with the corresponding effective pressure and the velocity of the liquid component along this axis, both of which are equal to zero. The flow is planar and the liquid rotates along the annulus with the velocity ptoportional to ζ_1. The reason for this is easily discerned when we recall that without the viscosity the liquid would not rotate relative to the inertial reference system but move as a unit with the velocity [r, ω] relative to the torus. The viscosity causes the outermost layers of the liquid to adhere to the torus surface and a drag develops between the torus and the liquid which results in a more complex distribution of velocities in the cross section of the torus.

Substituting the solution (3.2) into (1.1), we obtain

$$P_{zz} = \frac{16\pi\sqrt{2}}{15}\left[\frac{1}{2}\int_{\tau_0}^{\infty}\frac{f_0(\tau)}{\tau^2-1}Q^3_{-1/2}(\tau)\,d\tau + \sum_{m=1}^{\infty}\int_{\tau_0}^{\infty}\frac{f_m(\tau)}{\tau^2-1}Q^3_{m-1/2}(\tau)\,d\tau\right]$$

At large τ_0 the integrals can be easily computed to give

$$P_{zz} = \frac{\pi^2}{4\tau_0^4}\left[1 + O\left(\frac{1}{\tau_0^2}\right)\right] \tag{3.3}$$

4. Angular acceleration along the ox-axis. Let us now consider a more complicated case when $i = 1$ and $e^{(i)} = e_x$. (It is clear that the case $i = 2$ is completely analogous). Let us determine ζ_1. The difference between the system (2.1) for this case and (3.1) can be described by the following cyclic transformation $x \to y \to z \to x$. From this it is evident that $\zeta_{1x} \equiv 0$ and $\zeta_{1z} = \zeta_1 \sin\varphi$, where ζ_1 is given by the formulas (3.2). The solution for ζ_{1y} has the form

$$\zeta_{1y} = A^{1/2}\sum_{m=1}^{\infty} g_m(\tau)\sin m\beta \tag{4.1}$$

$$g_m(\tau) = c_m Q_{m-1/2}(\tau) + \frac{8\sqrt{2}}{15\pi}\frac{m}{\sqrt{\tau^2-1}}Q^1_{m+1/2}(\tau)$$

The coefficients c_m can be determined from the boundary condition $g_m(\tau_0) = 0$.

Let us find ζ_2. Using (4.1) we find that div $\zeta_1 = F$ and this leads to the following system for ζ_2:

$$\Delta\zeta_2 = \nabla s, \qquad \operatorname{div}\zeta_2 = -F, \qquad \zeta_2|_S = 0 \tag{4.2}$$

The boundary conditions are independent of φ and β, therefore the character of the dependence of ζ_2 and s on φ and β is determined only by the properties of the quantity F which is odd in β and contains φ only as $\sin\varphi$. As the result, ζ_2 and s can be written in the form

$$\zeta_{2x} = \frac{1}{2}\sin 2\varphi A^{1/2}\sum_{m=1}^{\infty} h_m(\tau)\sin m\beta \tag{4.3}$$

$$\zeta_{2y} = \frac{1}{2}A^{1/2}\sum_{m=1}^{\infty}[l_m(\tau) - \cos 2\varphi h_m(\tau)]\sin m\beta$$

$$\zeta_{2z} = A^{1/2}\sin\varphi\left[\frac{1}{2}q_0(\tau) + \sum_{m=1}^{\infty} q_m(\tau)\cos m\beta\right]$$

$$s = A^{1/2}\sin\varphi\sum_{m=1}^{\infty} a_m Q^1_{m-1/2}(\tau)\sin m\beta$$

The last equation of (4.3) takes into account the harmonic property of the functions s (see e.g. [2]). The coefficients a_m and the functions $h_m(\tau)$, $l_m(\tau)$ and $q_m(\tau)$ can obviously be found from the equations and the boundary conditions.

Let us write the functions h_m, l_m and q_m in the form resembling that of $f_m(\tau)$ and $g_m(\tau)$ (see (3.2) and (4.1), where d_m, e_m and k_m are constants)

$$h_m(\tau) = 2d_m Q^2_{m-1/2}(\tau) + H_m(\tau) \tag{4.4}$$
$$l_m(\tau) = 2e_m Q_{m-1/2}(\tau) + L_m(\tau)$$
$$q_m(\tau) = k_m Q^1_{m-1/2}(\tau) + K_m(\tau)$$

The first terms in these expressions make no contribution to the left-hand part of the first equation of (4.2). To clarify the manner in which d_m, e_m and k_m are chosen, we consider the following formal symmetry of Eqs. (4.2).

The coordinates x, y and z remain invariant under the substitution $\tau \to e^{i\pi}\tau$ and $\beta \to \pi - \beta$, but on the other hand, we have $F \to -F$ and $s \to -s$ (see (4.1) and (4.3)). Therefore we must have $\zeta_2 \to -\zeta_2$ so that e.g. $h_m \to \bar{e}^{\,i\pi/2}(-1)^m h_m$ (see 4.3)). The function $h_m(\tau)$ remains finite when $\tau \to \infty$, consequently it must have the form $h_m(\tau) \sim \tau^{-m-1/2}[1 + O(\tau^{-2})]$. The same results are obtained for $l_m(\tau)$ and $q_m(\tau)$. The properties of the Legendre functions [3] allow us to choose the coefficients d_m, e_m and k_m so that the second terms in (4.4) decrease for $\tau \to \infty$ more rapidly than the first terms. Let us expand these terms into series in Legendre functions of second kind, which form a complete, although not orthogonal system

$$H_m(\tau) = \sum_{n=0}^{\infty} \frac{8\sqrt{2}}{3\pi}(m + 2n + 2)\, H_m{}^n Q^2_{m+3/2+2n}(\tau)$$

$$L_m(\tau) = \sum_{n=0}^{\infty} \frac{8\sqrt{2}}{3\pi}(m + 2n + 2)\, L_m{}^n\, Q_{m+3/2+2n}(\tau)$$

$$K_m(\tau) = \sum_{n=0}^{\infty} \frac{8\sqrt{2}}{3\pi}(m + 2n + 2)\, K_m{}^n Q^1_{m+3/2+2n}(\tau)$$

Since the computations that follow are very cumbersome, we shall omit them, explaining just the general idea. We can, of course, seek the functions $h_m(\tau)$, $l_m(\tau)$ and $q_m(\tau)$ as well as the coefficients a_m, directly from (4.2). However, we find that it is simpler to replace the first equation of (4.2) by the equivalent property that rot ζ_2 is harmonic.

The harmonic property imposes stringent restrictions on the character of dependence of the functions on the toroidal variables τ, β and φ [2]. For this reason the three conditions of harmonicity of rot ζ_2 lead to three recurrent relations connecting $H_m{}^n$, $L_m{}^n$ and $K_m{}^n$. We note that only two of these conditions are independent by virtue of the identity div rot $\zeta_2 = 0$. It is convenient to employ a condition of harmonicity of div ζ_2 which yields another relation connecting $H_m{}^n$, $L_m{}^n$ and $K_m{}^n$. From (4.2) we can obtain for div ζ_2 the relations connecting $H_m{}^n$, $L_m{}^n$ and $K_m{}^n$ with d_m, e_m and k_m which contain the already determined coefficients b_m and c_m (see (3.2) and (4.1)).

Naturally, the conditions of harmonicity and the equations for div ζ_2 do not completely determine all unknown coefficients. We must also use the boundary conditions given in (4.2) which can be rewritten in the form

$$h_m(\tau_0) = 0, \quad l_m(\tau_0) = 0, \quad q_m(\tau_0) = 0 \tag{4.5}$$

Obviously, these conditions contain simultaneously the coefficients of $H_m{}^n$, $L_m{}^n$ and $K_m{}^n$ with all values of n, therefore we cannot determine them one after the other. The situation is simplified when τ_0 is assumed large and the unknown coefficients are sought in the form of series in $\tau_0{}^{-2}$. This particular choice follows from the fact that for large τ the Legendre functions can be expanded into series in τ^{-2} [3]. When the expansion is terminated at any stage, the conditions (4.5) contain a limited number of coefficients and form, together with the recurrent relations, a system of linear equations of finite order. Such a procedure enables us, in principle, to determine any of the coefficients with any required degree of accuracy.

Since the number of the coefficients is large, even low order approximations involve a considerable amount of manipulation. Below we give the first nonvanishing approximation for ζ in its explicit form, assuming that $\tau > \tau_0 \gg 1$

$$\zeta_x = \frac{1}{16} A^{1/2} \sin 2\varphi \sin\beta \frac{1}{\tau^{3/2}} \left(-\frac{1}{\tau_0^2} + \frac{1}{\tau^2}\right) \tag{4.6}$$

$$\zeta_y = \frac{1}{16} A^{1/2} \sin\beta (3 + \cos 2\varphi) \frac{1}{\tau^{3/2}} \left(\frac{1}{\tau_0^2} - \frac{1}{\tau^2}\right)$$

$$\zeta_z = \frac{1}{16} A^{1/2} \sin\varphi \frac{1}{\tau^{1/2}} \left[\left(\frac{1}{2\tau/^4} - \frac{3}{2}\frac{1}{\tau_0^2\tau^2} + \frac{1}{\tau^4}\right) + \cos\beta \frac{1}{\tau}\left(-\frac{1}{\tau_0^2} + \frac{1}{\tau^2}\right)\right]$$

The first approximation for P_{xx} which can be obtained by substituting (4.6) into (1.1), is

$$P_{xx} = \frac{\pi^2}{16\tau_0^6}\left[1 + O(\tau_0^{-2})\right] \tag{4.7}$$

5. Character of the motion of liquid. Effect of the liquid on the top. In the expressions (3.3) and (4.7) we have assumed that $c^2 = R^2 - r^2 = 1$. The explicit expressions for P_{xx} and P_{zz} in terms of the torus radii are

$$P_{xx} = P_{yy} = \frac{\pi^2}{16} Rr^6 \left[1 + O(r^2/R^2)\right], \qquad P_{zz} = \frac{\pi^2}{4} R^3 r^4 \left[1 + O(r^2/R^2)\right]$$

Clearly, $P_{xx} \ll P_{zz}$ at least when $r \ll R$. To discover the reason for this, we shall consider the qualitative aspects of the character of the motion of liquid in a torus under the action of various angular accelerations. In all the cases the motion of the liquid is vortical, but the vortices differ substantially from case to case.

When the angular acceleration is directed along the oz-axis, the vortex flows along the whole tube of the torus. The size of the vortex is determined by the median line radius R of the torus. The moment L is small for two reasons: (1) when the tube radius is reduced, so is its cross section, i.e. the mass of liquid within the vortex is also reduced, (2) adhesion of the liquid to the walls reduces its velocity in a narrow tube.

When the angular acceleration is directed along the ox-axis, the reasons given above are supplemented by another one which is the change in the character of motion. Near the ox-axis the vortices are plane, but further away they become curved due to the curvature of the walls and the Coriolis forces. The character of the dependence of ζ_x on φ shows that the liquid in motion does not intersect the plane $x = 0$. Thus each closed vortex is bounded only by a part of the tube and does not traverse its whole length. As the result, the distance between the layers of liquid moving in the opposite directions is determined by the small radius r of the tube. This leads to a sharp increase in the interlayer friction thus reducing the liquid velocity. Moreover, the fact that points moving with opposite velocities are near each other, leads to a further reduction in the contribution of the motion of the liquid towards the moment of impulse of the solid-fluid system. Therefore the result is $P_{xx} \ll P_{zz}$.

The above discussion appears to be qualitatively applicable to more complex configurations of the closed tube. Two types of flow can exist in any such tube, one extending throughout the whole tube, and the other through a part of it. When a close vortex extending through the whole tube is present, then the moment L is larger than in its absence. For a plane tube the moment, and therefore the value of P_{ii} is largest when the angular acceleration is perpendicular to the plane of the tube.

The moment of forces acted upon the top by the liquid has a retarding influence on the angular acceleration. Therefore the liquid contained in the cavity of the top exerts a stabilizing influence [1, 4]. In the case when the diagonal components of P_{ii} are different from each other, an optimal orientation of the cavity exists for which the stabilization time is shortest [4]. For a torus this situation arises when the principal axes with the largest and smallest moment of inertia are parallel to the plane of the torus.

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ADIABATIC SHOCK CURVE IN MAGNETIZABLE NONCONDUCTING MEDIA

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We investigate the relationships at the discontinuities in magnetizable nonconducting media. The magnetic permeability is assumed to be an arbitrary function of the magnetic field and, generally speaking, different on each side of the discontinuity. We note that the contribution of the terms connected with the magnetizability towards the relations at the discontinuity is substantial also in the case when the values of permeability at both sides of the discontinuity are constant and different from each other. We show that the behavior of the adiabatic shock curve depends substantially on the sign of the difference in the values of the permeability ahead and behind the discontinuity.